Staggered fermions

&

Power - counting

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Motivations

- Inductive proof of perturbative renormalizability.
- Ingredients:
  - PC Thm.
  - Compatible subtraction scheme.
  - Thm. that subtraction is equiv. to field, mass & coupl. renorm.

- The role of the PCThm is to establish that diagrams are finite, once subtractions implemented.
- Requirements:
  - Easily computed index that only depends on asymptotic properties of propagators & vertices.
  - Index computation should be inductive, relying on the indices of subdiagrams of lower order.

- Lattice subtlety: propagators & vertices periodic on reciprocal lattice $\frac{2\pi}{a}\mathbb{Z}^4$.
  - $\therefore$ Index based on large momentum makes no sense.
- Reisz's soln.: scaling of propagators & vertices w/ $a \Rightarrow \lambda a$.
- However: staggered fermions violate the conditions of Reisz's PC Thm.

- SFs are efficient to simulate; their mass is mult. renorm.'d → "chiral."

- 1-loop easy to power-count in "taste" bases.

- Higher loops, much harder.

- To date, no all-orders proof of PC Thm for SFs.

- Hence no rigorous proof of renormalizability, by pre-Wilsonian methods.

II. SF Pert. Th. vs. Reisz PC Thm.

(A) Reisz review

Partition domain of integration, depending on values of line momenta:

$$l_i = C_{ij} k_j + D_{ij} q_j$$

k loop, q ext., l line

$i \in \{1, \dots, N_l\}$

Take a subset $J \subseteq \{1, \dots, N_l\}$.

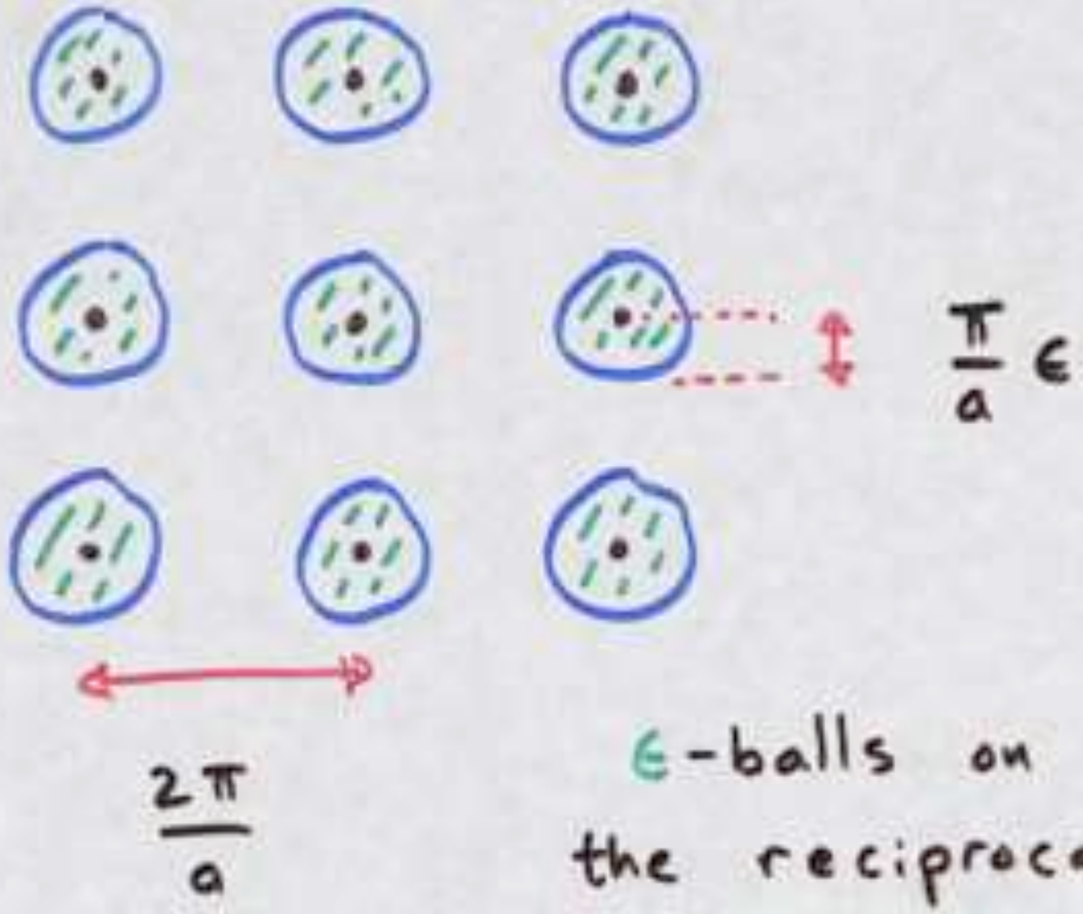
Let $\epsilon \ll 1$ and consider the domain of k where:

- For $i \in J$, ℓ_i is ϵ close to a pole in some Brillouin zone. I.e., $\exists$ $z_i \in \mathbb{Z}^4$ s.t.

$$\left\| \ell_i - \frac{2\pi}{a} z_i \right\| \leq \frac{\pi}{a} \epsilon \quad (*)$$

- For $i \notin J$, ℓ_i is ϵ far from all poles. I.e., $\nexists$ $z_i \in \mathbb{Z}^4$ s.t. $(*)$ holds. Rather, $\forall$ $z_i \in \mathbb{Z}^4$

$$\left\| \ell_i - \frac{2\pi}{a} z_i \right\| > \frac{\pi}{a} \epsilon .$$



ϵ-balls on the reciprocal lattice $\frac{2\pi}{a}\mathbb{Z}^4$

ϵ - resolution of identity

$$\Theta_\epsilon(\ell) = \begin{cases} 0 & \text{if } \ell \text{ falls in a ball} \\ 1 & \text{if } \ell \text{ doesn't} \end{cases}$$

$$\mathbb{1} = \Theta_\epsilon(\ell) + \sum_z \Theta\left(\frac{\pi}{a}\epsilon - \left\|\ell - \frac{2\pi}{a} z\right\|\right)$$

- Stick in $\mathbb{1}$ for each line momentum ℓ_i.
- This partitions the Feynman integral.

$$\hat{I} = \sum_{J,z} \hat{I}_{Jz}$$

Then for any "reasonable" lattice propagator

$$\frac{N(l_i; m, a)}{\zeta(l_i; m, a)}$$

the following bounds hold:

- $i \in J$: $(l_i \in \mathcal{B}_a^{(1)})$

$$\zeta(l_i; m, a)^{-1} \leq (\text{const.})(l_i^2 + m^2)^{-1}$$

- $i \notin J$:

$$\zeta(l_i; m, a)^{-1} \leq (\text{const.})\, a^2$$

The denominator of a Feyman integrand is bounded by the following:

$$(a^2)^{N(i \notin J)} \prod_{i \in J} (l_i^2 + m^2)^{-1} \cdot (\text{const.})$$

Note that this is a rational function of k, q. The a dependence of the denominator factors out of the Feynman integral.

Caveats

(1) Numerators × Vertices require further trickery.

(2) $\pi\epsilon/a$ – dependent domains of loop momenta k integration require an extension of continuum PC index.

(3) (THE BIGGIE) The bounds on p. 7

$$\zeta_{i\in J}^{-1} \lesssim (\ell_i^2+m^2)^{-1}, \quad \zeta_{i\notin J}^{-1} \lesssim a^2$$

assume

(A) 1st Brillouin zone for $i \in J$:

$$\ell \in \mathcal{B}^{(1)} = \left(-\frac{\pi}{a}, \frac{\pi}{a}\right]^4$$

But can show that for small enough a, ϵ, $\exists$

$$\Delta_j \in \frac{2\pi}{a} \mathbb{Z}^4 \quad \text{s.t.}$$

$$k_j \to k_j + \Delta_j \quad (j = 1, \dots, N_k)$$

brings $\ell_i \in \mathcal{B}^{(1)} \ \forall \ i \in J$.

(B) The Feynman rules are assumed to be periodic on the reciprocal lattice $\frac{2\pi}{a}\mathbb{Z}^4$

and line momenta are assumed natural (in part, $l_i = C_{ij} k_j + D_{ij} q_j$ with $C_{ij} \in \mathbb{Z}$) so that the shift $k_j \to k_j + \Delta_j$,

$\Delta_j \in \frac{2\pi}{a}\mathbb{Z}^4$, used in (A), leaves form of Feynman integrand unchanged.

(C) [The one SFs violate]

For $l \in \mathcal{B}^{(1)}$, the only pole* is at $l \cong 0$.

*pole = local min. of denom. G, $\lesssim \mathcal{O}(m^2)$.

$$\left.\begin{array}{l} C_F = a^{-2} \sum_\mu \sin^2(l_\mu a) + m^2 \\[2ex] l_\mu \in \frac{\pi}{a} \mathbb{Z}^4 \qquad \text{poles} \end{array}\right\} SF_3$$

- There are 15 poles that are ϵ-far from $\frac{2\pi}{a}\mathbb{Z}^4$ but still in the first Brillouin zone $\mathcal{B}^{(1)}$.

- For these, the bound

$$C^{-1}_{i \notin J} \leq (\text{const.})\, a^2$$

fails.

III. Do I taste a way out?

- What I have just said is well-known. It was pointed out long ago (1988) by Lüscher.

- "But," I asked myself, "what about taste bases?"

- There, poles in $\mathcal{B}^{(1)}$ are reinterpreted as spin-taste labels!

$$K \equiv \{ (0^4), (\underline{1,0^3}), (\underline{1^2,0^2}), (\underline{1^3,0}), (1^4) \}$$

$$K^* \equiv K - \{ (0^4) \}$$

- The 15 extra poles are at:

$$n \in K^*$$

$$l = n \frac{\pi}{a}$$

- Sub. into 1-component SF propagator:

$$S\left(p = p_r + \frac{\pi}{a}A,\ q = q_r + \frac{\pi}{a}B\right)$$

$$\equiv S_{AB}(p_r, q_r) = \frac{N_{AB}}{C}$$

$A, B \in K \qquad p, q \in \mathcal{B}^{(1)}_a$

$p_r, q_r \in \mathcal{B}^{(1)}_{2a} = \left(\frac{\pi}{2a}, \frac{\pi}{2a}\right]^4$

reduced Brillouin zone

$\delta(p_r + q_r)\cdot$

$$N_{AB} = m\,(\overline{\overline{1 \otimes 1}})_{AB} - i a^{-1} \sum_\mu \sin(p_{r\mu} a)(\overline{\overline{\gamma_\mu \otimes 1}})_{AB}$$

$$C = a^{-2} \sum_\mu \sin^2(p_{r\mu} a) + m^2$$

Unique pole for $p_r \in \mathcal{B}^{(1)}_{2a}$

is $p_r = 0$.

Does that solve the problem?

- Try to apply Reisz's method.

- Identify line momenta that are ϵ-close to one of the poles of G.

- Introduce ϵ-balls$_F$ on reduced reciprocal lattice $\frac{\pi}{a}\mathbb{Z}^4$.

$$\Theta_\epsilon^F(\ell) = \begin{cases} 0 & \text{if } \ell \text{ fall in a ball}_F \\ 1 & \text{if } \ell \text{ doesn't} \end{cases}$$

$$\mathbb{1} = \Theta_\epsilon^F + \sum_{\substack{z \in \mathbb{Z}^4 \\ \eta \in K}} \Theta\left(\frac{\pi}{a}\epsilon - \left\| \ell - \frac{2\pi}{a} z - \frac{\pi}{a}\eta \right\|\right)$$

Important point

- In general, $\not\exists$

  $\Delta_j \in \frac{2\pi}{a} \mathbb{Z}^4$, $k_j \to k_j + \Delta_j$

  s.t. $l_{i \in J_F}$ are shifted to $\mathcal{B}^{(1)}_{2a}$.

- However, $\exists$ (always)

  $\Delta_j \in \frac{\pi}{a} \mathbb{Z}^4$, $k_j \to k_j + \Delta_j$

  s.t. l_i are shifted to $\mathcal{B}^{(1)}_{2a}$, for small enough a, ϵ.

- In the latter case, the bound

  $$G^{-1} \leq (\text{const.}) (l_i^2 + m^2)^{-1}$$

  holds.

- However, the Feynman rules are only $\frac{2\pi}{a}$-periodic, due to gluons.
- The shift

$$k_j \to k_j + \Delta_j \,, \quad \Delta_j \in \frac{\pi}{a}\mathbb{Z}^4$$

is not an invariance of the integral. It is not legal.

- Note that this is not a change of integration variables. We keep the domain $k_j \in \mathcal{B}^{(1)}_{2a}$ fixed.

It is not a full period.

IV. Projection

$$\pi : \mathbb{R}^4 \to \mathcal{B}_{2a}^{(1)}$$

$$\pi(\ell) = \ell_r \in \mathcal{B}_{2a}^{(1)}$$
$$= \ell - \frac{\pi}{a} z, \quad z \in \mathbb{Z}^4.$$

- For $\ell_i \in J_F$ (ϵ-close to ball_F)

$$G_i^{-1} \leq (\text{const.}) \left(\pi(\ell_i)^2 + m^2\right)^{-1}$$

- Denom. bound:

$$(a^2)^{N(i \notin J_B) + N(i \notin J_F)}$$

$$\cdot \prod_{i \in J_B} (\ell_i^2 + \lambda^2)^{-1} \cdot \prod_{i \in J_F} \left(\pi(\ell_i)^2 + m^2\right)^{-1}$$

- Why not just use this?
- B/c factor

$$\prod_{i \in J_F} \left(\left(l_i - \frac{\pi}{a} z_i \right)^2 + m^2 \right)^{-1}$$

$$l_i = C_{ij} k_j + D_{ij} q_j$$

$$z_i \in \mathbb{Z}^4$$

is not natural, does not have a nice continuum interpretation.

- I.e., cannot use continuum PC Thm due to $\mathcal{O}(\pi/a)$ terms.
- It is like $\mathcal{O}(\pi/a)$ external momentum flowing in from nowhere.